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Finite Quantum Tomography and Semidefinite Programming

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Using the convex semidefinite programming method and superoperator formalism we obtain the finite quantum tomography of some mixed quantum states such as: truncated coherent states tomography, phase tomography and coherent spin state tomography, qudit tomography, N-qubit tomography, where that obtained results are in agreement with those of References (Buzek *et al.*, *Chaos, Solitons and Fractals* **10** (1999) 981; Schack and Caves, Separable states of N quantum bits. In: *Proceedings of the X. International Symposium on Theoretical Electrical Engineering*, **73**. W. Mathis and T. Schindler, eds. Otto-von-Guericke University of Magdeburg, Germany (1999); Pegg and Barnett *Physical Review A* **39** (1989) 1665; Barnett and Pegg *Journal of Modern Optics* **36** (1989) 7; St. Weigert *Acta Physica Slov.* **4** (1999) 613).

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1. INTRODUCTION

The quantum complementarity principle does not allow to recover the quantum state from measurements on a single system, unless we have some prior information on it. On the other hand, the no cloning theorem ensures that it is not possible to make exact copies of a quantum system, without having prior knowledge of its state. Hence, the only possibility for devising a state reconstruction procedure is to provide a measuring strategy that employs numerous identical (although unknown) copies of the system, so that different measurements may be performed on each of the copies.

The problem of state estimation resorts essentially to estimating arbitrary operators of a quantum system by using the result of measurements of a set of

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observables. If this set of observables is sufficient to give full knowledge of the system state, then we define it as a quorum. Notice that, in general, a system may allow various, different quorums. Quantum tomography that was born (Raymer and Schleich, 1997; McAlister and Raymer, 1997) as a state reconstruction technique in the optical domain, makes use of the results of the quorum measurements in order to reconstruct the expectation value of arbitrary operators (even not observables) acting on the system Hilbert space.

On the other hand, a precise knowledge of the density matrix would require an infinite number of measurements on identical preparations of radiation. However, in real experiments one has only a finite number of data at ones disposal, and thus a statistical analysis and errors estimation are needed.

Authors of Buzek *et al.* (1999) presented several schemes for a reconstruction of states of quantum systems from measured data:

(1) The maximum entropy (MaxEnt) principle can be efficiently used for an estimation of quantum states (i.e., density operators or Wigner functions) on incomplete observation levels, when just a fraction of system observables are measured (i.e., the mean values of these observables are known from the measurement). In the limit, when all system observables (i.e., the quorum of observables) are measured, the MaxEnt principle leads to a complete reconstruction of quantum states, i.e. quantum states are uniquely determined.

(2) When only a finite number of identically prepared systems are measured, then the measured data contain only information about frequencies of appearances of eigenstates of certain observables. They showed that in this case states of quantum systems can be estimated with the help of quantum Bayesian inference.

(3) They showed how to construct the optimal generalized measurement of a finite number of identically prepared quantum systems which results in the estimation of a quantum state with the highest fidelity and showed how this optimal measurement can in principle be realized. They analyzed two physically interesting examples—a reconstruction of states of a spin-1/2 and an estimation of phase shifts (Buzek *et al.*, 1999).

On the other hand, over the past years, semidefinite programming (SDP) has been recognized as valuable numerical tools for control system analysis and design. In (SDP) one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. SDP, has been studied (under various names) as far back as the 1940s. Subsequent research in semidefinite programming during the 1990s was driven by applications in combinatorial optimization (Goemans and Williamson, 1995), communications and signal processing (Luo, 2003; Davidson *et al.*, 2000; Ma *et al.*, 2002), and other areas of engineering (Boyd *et al.*, 1994). Although semidefinite programming is designed to be applied in numerical methods it can be used for analytic computations, too. Some authors try to use the SDP to construct an explicit entanglement witness (Doherty *et al.*, 2002; Parrilo *et al.*, 2002). Kitaev used semidefinite programming duality to prove the impossibility of quantum coin flipping (Kitaev, 2002), and Rains gave bounds on distillable entanglement using semidefinite programming (Rains, 2001). In the context of quantum computation, Barnum, *et al.*, reformulated quantum query complexity in terms of a semidefinite program (Barnum *et al.*, 2003). The problem of finding the optimal measurement to distinguish between a set of quantum states was first formulated as a semidefinite program in 1972 by Holevo, who gave optimality conditions equivalent to the complementary slackness conditions (Helstrom, 1976). Recently, Eldar *et al.*, showed that the optimal measurements can be found efficiently by solving the dual followed by the use of linear programming (Eldar *et al.*, 2003). Also in Lawrence (2003) used semidefinite programming to show that the standard algorithm implements the optimal set of measurements. All of the above mentioned applications indicate that the method of SDP is very useful.

In a laboratory and in practice, we always deal with finite ensembles of copies of the measured system. This implies the need of developing novel tools specially designed to process realistic and finite experimental samples. Then it is necessary to truncate the Hilbert space to a FD basis (Buzek and Drobny, 2000). In this paper we use the SDP method in order to obtain quantum tomography with truncating the infinite Banach space to a FD basis.

The paper is organized as follows:
In Section 2 we define semidefinite programming. In Section 3 we define superoperator formalism. In Section 4 we describe the projection method and using SDP method and superoperator formalism obtain finite quantum tomography. In Section 5 we obtain truncated generalized coherent states quantum tomography with semidefinite programming, In Section 5 we obtain some typical finite quantum tomographic examples, such as: finite dimensional phase tomography and coherent spin state tomography, finite dimensional qudit quantum tomography, N-qubit tomography, with SDP method and superoperator formalism. The paper is ended with a brief conclusion.

2. SEMI-DEFINITE PROGRAMMING

A SDP is a particular type of convex optimization problem (Vandenberghe and Boyd, 1996). A SDP problem requires minimizing a linear function subject to a linear matrix inequality (LMI) constraint (Vandenberghe and Boyd, unpublished):

$$\begin{aligned} &\text{minimize} \quad \mathcal{P} = c^T x \\ &\text{subject to} \quad F(x) \geq 0, \end{aligned} \tag{2.1}$$

where c is a given vector, $x^T = (x_1, \ldots, x_n)$, and $F(x) = F_0 + \sum_i x_i F_i$, for some fixed hermitian matrices F_i. The inequality sign in $F(x) \geq 0$ means that $F(x)$ is positive semidefinite.

This problem is called the primal problem. Vectors x whose components are the variables of the problem and satisfy the constraint $F(x) \geq 0$ are called primal feasible points, and if they satisfy $F(x) > 0$ they are called strictly feasible points. The minimal objective value $c^T x$ is by convention denoted as $\mathcal{P}^*$ and is called the primal optimal value.

Due to the convexity of set of feasible points, SDP has a nice duality structure, with, the associated dual program being:

$$\begin{aligned} \text{maximize} \quad & -Tr[F_0 Z] \\ & Z \geq 0 \\ & Tr[F_i Z] = c_i. \end{aligned} \tag{2.2}$$

Here the variable is the real symmetric (or Hermitean) matrix Z, and the data c, F_i are the same as in the primal problem. Correspondingly, matrices Z satisfying the constraints are called dual feasible (or strictly dual feasible if $Z > 0$). The maximal objective value $-Tr F_0 Z$, the dual optimal value, is denoted as d^*.

The objective value of a primal(dual) feasible point is an upper (lower) bound on $\mathcal{P}^*(d^*)$. The main reason why one is interested in the dual problem is that one can prove that $d^* \leq \mathcal{P}^*$, and under relatively mild assumptions, we can have $\mathcal{P}^* = d^*$. If the equality holds, one can prove the following optimality condition on x:

A primal feasible x and a dual feasible Z are optimal which is denoted by $\hat{x}$ and $\hat{Z}$ if and only if

$$F(\hat{x})\hat{Z} = \hat{Z}F(\hat{x}) = 0. \tag{2.3}$$

This latter condition is called the complementary slackness condition.

In one way or another, numerical methods for solving SDP problems always exploit the inequality $d \leq d^* \leq \mathcal{P}^* \leq \mathcal{P}$, where d and $\mathcal{P}$ are the objective values for any dual feasible point and primal feasible point, respectively. The difference

$$\mathcal{P} - d = c^T x + Tr[F_0 Z] = Tr[F(x)Z] \geq 0 \tag{2.4}$$

is called the duality gap. If the equality $d^* = \mathcal{P}^*$ holds, i.e., the optimal duality gap is zero, then we say that strong duality holds.

3. SUPEROPERATOR FORMALISM

In order to treat discrete and continuous density operator representations on an equal footing, we introduce the following superoperator formalism. The set of linear operators acting on a D-dimensional Hilbert space H is a D^2-dimensional complex vector space $\mathcal{L}(\mathcal{H})$. Let us introduce operator "kets" $\mid A) = A$ and "bras" $(A \mid = A^\dagger$, distinguished from vector kets and bras by the use of round brackets. Then the natural inner product on $L(H)$, the trace-norm inner product, can be written as $(A \mid B) = tr(A^{\dagger}B)$. The notation $S =\mid A)(B \mid$ defines a superoperator S acting like

$$S \mid X) =\mid A)(B \mid X) = tr(B^{\dagger}X)A. \tag{3.5}$$

Now let the set $\{\mid N_j)\}$ constitute a (complete or overcomplete) operator basis; i.e., let the operator kets $\mid N_j)$ span the vector space $L(H)$. It follows that the superoperator $\mathcal{G}$ defined by

$$\mathcal{G} \equiv \sum_j \mid N_j)(N_j \mid \tag{3.6}$$

is invertible. The operators

$$Q_j \equiv \mathcal{G}^{-1} \mid N_j) \tag{3.7}$$

form a dual basis, which gives rise to the following resolutions of the superoperator identity:

$$1 = \sum_j \mid Q_j)(N_j \mid = \sum_j \mid N_j)(Q_j \mid . \tag{3.8}$$

An arbitrary operator A can be expanded as

$$A = \sum_j \mid N_j)(Q_j \mid A) = \sum_j N_j tr(Q_j^{\dagger}A) \tag{3.9}$$

and

$$A = \sum_j \mid Q_j)(N_j \mid A) = \sum_j Q_j tr(N_j^{\dagger}A). \tag{3.10}$$

These expansions are unique if and only if the operators N_j are linearly independent (Schack and Caves, 1999).

4. PROJECTION METHOD AS A SEMIDEFINITE PROGRAMMING AND FINITE QUANTUM TOMOGRAPHY

4.1. Bases and Frames

In this section we collect some rudimentary facts that will be used in what follows.

A *basis* is one of the most fundamental concepts in linear algebra.

A set of linearly independent vectors $\{e_i\}_{i=1}^n$ in a FD complex vector space V is a *basis* for V if, for each $f \in V$, there exist coefficients $c_1, c_2, \ldots, c_n \in \mathcal{C}$ such that

$$f = \sum_{i=1}^{n} c_i e_i. \tag{4.11}$$

The independence condition implies that the coefficients $c_1, \ldots, c_n$ are unique.

For infinite dimensional (ID) vector spaces, the concept of a basis is more complicated.

An $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is an *orthonormal system* (ONS) Christensen and Jensen (1999) if

$$\langle e_i, e_j \rangle = \delta_{ij}. \tag{4.12}$$

An ONS $\{e_i\}_{i=1}^{\infty}$ is an *orthonormal basis* (ONB) if

$$\mathcal{H} = s\bar{p}an\{e_i\}_{i=1}^{\infty} \tag{4.13}$$

when $\{e_i\}_{i=1}^{\infty}$ is an ONB, each $f \in \mathcal{H}$ can be written as

$$f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i. \tag{4.14}$$

Definition. Two sequences $\{x_i\}$ and $\{y_i\}$ in a Hilbert space $\mathcal{H}$ are said to be *biorthonormal*, if

$$\langle x_i, y_j \rangle = \delta_{ij}. \tag{4.15}$$

A sequence $\{y_i\}$ biorthogonal to a basis $\{x_i\}$ for $\mathcal{H}$ is itself a basis for $\mathcal{H}$, and we have for each x the representation

$$x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i, \quad \text{and } x = \sum_{i=1}^{\infty} \langle x, x_i \rangle y_i. \tag{4.16}$$

Frame: A family of elements $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B||f||^2, \quad \forall f \in \mathcal{H}, \tag{4.17}$$

where I is a countable index set. The numbers A, B are called frame bounds. They are not unique. The optimal frame bounds are the biggest possible value for A and the smallest possible value for B in (4.17). If we can choose $A = B$, the frame is called tight. If a frame ceases to be a frame when any element is removed, the frame is said to be exact. Since a frame $\{f_i\}_{i \in I}$ is a Bessel sequence, the operator

$$T: l^2(I) \to \mathcal{H}, \quad T\{c_{i\ i \in I}\} = \sum_{i \in I} c_i f_i, \tag{4.18}$$

is bounded and linear; T is sometimes called the preframe operator. The adjoint operator is given by

$$T^*: \mathcal{H} \to l^2(I), \quad T^* f = \{\langle f, f_i \rangle\}_{i=1}^{\infty}. \tag{4.19}$$

By composing the operators T and T^*, we obtain the operator

$$S: \mathcal{H} \to \mathcal{H}, \quad Sf = TT^* f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i, \tag{4.20}$$

where S is called the frame operator with

$$AI \leq S \leq BI. \tag{4.21}$$

The frame operator is a bounded, positive, and invertible operator.

Proposition. *Let $\{f_i\}_{i \in I}$ be a frame. Then, the optimal bounds are given by Christensen and Jensen (1999)*

$$A = ||S^{-1}||^{-1}, \quad B = ||S||. \tag{4.22}$$

4.2. Frames in Finite-Dimensional Spaces

We investigate the properties of a frame generated by a finite subset of a Hilbert space.

Calculation of the frame coefficients $\{\langle f, S^{-1} f_i \rangle\}$ involves inversion of the frame operator S. In practice it can be a problem if the underlying Hilbert space is infinite dimensional. There is an approach to the problem as follows (Christensen and Jensen, 1999):

Given the frame $\{f_i\}_{i=1}^{\infty}$ we consider finite subsets $\{f_i\}_{i=1}^{n}$, $n \in N$. It can be shown that $\{f_i\}_{i=1}^{n}$ is a frame for $\mathcal{H}_n = span\{f_i\}_1^n$ and the corresponding frame operator is $S_n: \mathcal{H}_n \to \mathcal{H}_n$ and the orthogonal projection P_n on $\mathcal{H}_n$ is

$$P_n f = \sum_{i=1}^{n} \left\langle f, S_n^{-1} f_i \right\rangle f_i, \quad f \in \mathcal{H}. \tag{4.23}$$

For $n \to \infty$, $P_n f \to f = \sum_{i=1}^{\infty} \langle f, S^{-1} f_i \rangle f_i$, one can hope that the coefficients $\langle f, S_n^{-1} f_i \rangle$ converges to the frame coefficients for f, i.e., that

$$\left\langle f, S_n^{-1} f_i \right\rangle \to \langle f, S^{-1} f_i \rangle \ \text{ as } n \to \infty, \quad \forall i \in I, \quad \forall f \in \mathcal{H}. \tag{4.24}$$

If (4.24) is satisfied we say that the projection method works. In this case the frame coefficients can be approximated as close as we want using FD methods, i.e., linear algebra, since S_n is an operator on the FD space $\mathcal{H}_n$. This is a very important property for applications: for example, it makes it possible to use computers to approximate the frame coefficients.

Proposition 4.1. *Let $\{f_i\}_{i \in I}$ be a frame and that $I_n \nearrow I$ is an increasing family of finite index sets. Then $S_n^{-1} f_i \to S^{-1} f_i$ weakly as $n \to \infty$ for all $i \in I$ if and* *only if for any $i \in I$ there exists a constant c_i such that*

$$\left\| S_n^{-1} f_i \right\| \leq c_i \quad \forall n \text{ such that } i \in I_n \tag{4.25}$$

A frame $\{f_i\}$ is a Riesz basis if and only if it is finitely independent, i.e., any finite subset is linearly independent, and there exists an increasing family of finite index set $I_n \nearrow I$ such that $S_n^{-1} f_i \to S^{-1} f_i$ weakly as $n \to \infty$ for all $i \in I$ (Kim and Lim, 1997).

Our results are must conveniently formulated in operator terminology. Let $S, \{S_n\}_{n=1}^{\infty}$ be operators from $\mathcal{H}$ into the Hilbert space $(\mathcal{K}, \langle *, * \rangle_{\mathcal{K}})$. We say that $S_n \to S$ in strong operator topology if $S_n f \to Sf$, $\forall f \in \mathcal{H}$, and that $S_n \to S$ in the weak operator topology if $\langle g, S_n f \rangle_{\mathcal{K}} \to \langle g, Sf \rangle_{\mathcal{K}}$, $\forall f \in \mathcal{H}$ and, $\forall g \in \mathcal{K}$.

Proposition 4.2. *The strong projection method works for every Riesz frame (Casazza* et al.*, 2003).*

4.3. Projection Method and Semidefinite Programming

Now, using semidefinite programming we study the projection method.

At first we suppose that bases is orthogonal in infinite dimensional Hilbert space.

The corresponding frame operator is tight frame with bounds $A = B = 1$, i.e., we have:

$$S = \sum_{i=1}^{\infty} |e_i\rangle\langle e_i| = I. \tag{4.26}$$

We will try to approximate it with a frame operator in n-dimensional Hilbert space. i.e.,

$$S_n = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|. \tag{4.27}$$

It is easy to see that every finite collection of elements in $\mathcal{H}$ is a frame for its span.

According to SDP and complementary slackness we have

$$\hat{Z}(S - S_n) = \hat{Z}\left(I - \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|\right) = 0, \tag{4.28}$$

or equivalently,

$$\hat{Z}(S - S_n) = \hat{Z}\left(\sum_{i=1}^{n}(1 - \lambda_i)|e_i\rangle\langle e_i| + \sum_{i=n+1}^{\infty} |e_i\rangle\langle e_i|\right) = 0. \tag{4.29}$$

Now, taking its matrix element between $|\psi_j\rangle$, $1 \leq j \leq n$ we get

$$\langle \psi_j|\hat{Z}|\psi_j\rangle(1 - \lambda_j) = 0 \Rightarrow \lambda_j = 1, \quad 1 \leq j \leq n \tag{4.30}$$

and if $n+1 \leq j \leq \infty$, $\langle \psi_j | \hat{Z} | \psi_j \rangle = 0$ and therefore we conclude $|\psi_j\rangle \in ker \hat{Z}$. Finally the frame operator can be defined as

$$S_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|.$$

Now, if we repeat the above calculation for any operator in the infinite dimensional space we will get this operator in the FD space (projected operator).

If states be non-orthonormal, in this case we have

$$\hat{Z}\left(\sum_{i=1}^{\infty} |x_i\rangle\langle x_i| - \sum_{i=1}^{n} \lambda_i |x_i\rangle\langle x_i|\right) = 0. \tag{4.31}$$

Again, taking its matrix element between $|x_j\rangle, |y_j\rangle$ $1 \leq j \leq n$ where $|y_j\rangle$ is the dual state, we get

$$\langle x_j | \hat{Z}\left(\sum_{i=1}^{n} (1-\lambda_i)|x_i\rangle\langle x_i| + \sum_{i=n+1}^{\infty} |x_i\rangle\langle x_i|\right) |y_j\rangle = 0, \tag{4.32}$$

then

$$\langle x_j | \hat{Z} | x_j \rangle (1-\lambda_j) = 0 \Rightarrow \lambda_j = 1, \quad 1 \leq j \leq n, \tag{4.33}$$

and again for $n+1 \leq j \leq \infty$ we conclude that $|x_j\rangle \in ker \hat{Z}$ obviously the frame operator is defined as

$$S_n = \sum_{i=1}^{n} |x_i\rangle\langle x_i|.$$

which is a projection operator.

If we repeat the above calculation for frame, we obtain $S_n = \sum_{i=1}^{n} |f_i\rangle\langle f_i|$.

In the following section we find the finite quantum tomography using semidefinite programming.

4.4. Finite Quantum Tomography Via Semidefinite Programming

Quantum state reconstruction schemes can be understood as an a posterior estimation of density operator of a given quantum mechanical system based on data obtained with the help of a macroscopic measurement apparatus. Only if an infinite ensemble is given can one find out the state. But infinite ensembles don't exist in practice. In a laboratory and in practice, we always deal with finite ensembles of copies of the measured system. This implies the need of developing novel tools specially designed to process realistic and finite experimental samples. Then it is necessary to truncate the Hilbert space to a FD basis (Buzek and Drobny, 2000).

Now in this work using the projection method and semidefinite programming we express the mathematical structure correspond to finite tomography and obtain the tomographic formula based on finite Hilbert space.

At first, from (3.9) or (3.10) we assume that

$$\rho = \sum_j \mid Q_j)(N_j \mid \rho) = \sum_j Q_j tr(N_j^\dagger \rho), \tag{4.34}$$

is a density matrix in infinite dimensional Banach space, where $\{N_j\}$ constitute a operator basis in superoperator formalism. Also let

$$\rho^n = \sum_{j=1}^{n} \lambda_j \mid N_j), \tag{4.35}$$

be a density matrix in FD banach space which is obtained from truncating the ID Banach space.

Using the properties of density matrix we have

$$\rho - \rho^n \geq 0, \tag{4.36}$$

which in comparison with semidefinite programming we get

$$F_0 = \rho, \quad F_j =\mid N_j) \text{ and } x_j = \lambda_j, \quad \text{for} \quad j = 1, \ldots, n.$$

If we use the complementary slackness condition, and for a feasible $(\hat{Z}, \lambda_{j_{\max}})$, for $j = 1, \ldots, n$, we have

$$\hat{Z}(\rho - \rho^n) = 0, \tag{4.37}$$

or

$$\hat{Z}\left(\rho - \sum_{j=1}^{n} \lambda_j \mid N_j)\right) = 0. \tag{4.38}$$

Using resolution of the superoperator identity (3.8) we obtain

$$\sum_i \hat{Z} \mid N_i)(Q_i \mid \left[\rho - \sum_j \lambda_j \mid N_j)\right] = 0, \text{ for } j = 1, \ldots, n \tag{4.39}$$

Therefore, we have

$$\sum_i (\hat{Z} \mid N_i)[(Q_i \mid \rho) - \lambda_i] = 0, \quad i = 1, \ldots n. \tag{4.40}$$

It is obvious that $(\hat{Z} \mid N_i) = 0$ for $i > n$ then we conclude that $\mid N_i) \in ker \hat{Z}$. Then we obtain

$$\lambda_i = (Q_i \mid \rho) = tr[\rho Q_i^\dagger]. \tag{4.41}$$

Therefore we obtain the tomography formula in FD Banach space as the follow:

$$\rho^n = \sum_{i=1}^{n} \mid N_i)(Q_i \mid \rho) = \sum_{i=1}^{n} \mid N_i) tr[\rho Q_i^{\dagger}]. \tag{4.42}$$

In the following, we will consider density matrix with orthogonal states of the form:

$$\rho = \sum_{j}^{\infty} tr(\rho|\psi_j\rangle\langle\psi_j|)|\psi_j\rangle\langle\psi_j|$$

where is a density matrix in ID Hilbert space. In the superoperator formalism we can write

$$\mid N_j) = \mid Q_j) = |\psi_j\rangle\langle\psi_j| \tag{4.43}$$

Also let

$$\rho^n = \sum_{j=1}^{n} \lambda_j \mid N_j) = \sum_{j=1}^{n} \lambda_j |\psi_j\rangle\langle\psi_j|$$

be a density matrix in FD Hilbert space which is obtained from truncating the infinite dimensional Hilbert space and $|\psi\rangle$ is an orthogonal state.

Using (4.42) we obtain the tomography formula in finite dimensional Hilbert space as the follow:

$$\rho^n = \sum_{j=1}^{n} tr(\rho|\psi_j\rangle\langle\psi_j|)|\psi_j\rangle\langle\psi_j|. \tag{4.44}$$

In the following we describe some examples for FD quantum tomogarphy.

5. TRUNCATED COHERENT STATES QUANTUM TOMOGRAPHY WITH SEMIDEFINITE PROGRAMMING

Quantum homodyne tomography is used in quantum optic in the measurement of the quantum state of light (Glauber Cs). In this case, we get (D'Ariano, 1990; D'Ariano *et al.*, 2001):

$$\hat{\rho} = \int_{\mathcal{C}} \frac{d^2\alpha}{\pi} Tr[\hat{\rho}\hat{U}^{\dagger}(\alpha)]\hat{U}(\alpha), \tag{5.45}$$

where $\hat{U}(\alpha) = \exp(\alpha a^{\dagger} - \alpha^* a)$ is a displacement operator.

For the complex Fourier transform of the displacement operator $\hat{U}$ (D'Ariano *et al.*, 1996)

$$\hat{U}(\alpha) = \int \frac{d^2\xi}{\pi} \hat{U}(\xi) \exp(\alpha\xi^* - \alpha^*\xi), \tag{5.46}$$

the expansion of the operator in terms of the operator $\hat{U}(\alpha)$ is given by

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} W(\alpha)\hat{U}(\alpha), \tag{5.47}$$

where $W(\alpha)$ is Wigner function.

In Miquel *et al.* (2002) formalism was applied to represent the states and the evolution of a quantum system in phase space in FD Hilbert space and, finally, it was discussed how to perform direct measurement to determine the wigner function. This approach was based on the use of phase space point operator to define Wigner function. For discrete systems we can define finite translation operators $\hat{Q}$ and $\hat{V}$, which generate finite translation in position and momentum, respectively. The translation operator $\hat{Q}$ generates cyclic shifts in the position basis and is diagonal in momentum basis:

$$\hat{Q}^m \mid n\rangle = \mid n+m\rangle, \quad \hat{Q}^m \mid k\rangle = \exp(-2\pi imk/N) \mid k\rangle. \tag{5.48}$$

Similarly, the operator $\hat{V}$ is a shift in the momentum basis and is diagonal in position basis:

$$\hat{V}^m \mid k\rangle = \mid k+m\rangle, \quad \hat{V}^m \mid n\rangle = \exp(2\pi imn/N) \mid n\rangle. \tag{5.49}$$

Now by identifying the corresponding displacement operators, the discrete analogue of the phase space translation operator is given by:

$$\hat{U}(q, p) = \hat{Q}^q \hat{V}^p \exp[(i\pi pq/N)]. \tag{5.50}$$

Here we can define the point operator as:

$$\hat{A}(q, p) = \frac{1}{(2N)^2} \sum_{n,m=0}^{2N-1} \hat{\pi}(m, k) \exp\left(-2\pi i \frac{(kq - mp)}{2N}\right), \tag{5.51}$$

or as:

$$\hat{A}(\alpha) = \frac{1}{2N} \hat{Q}^q \hat{R} \hat{V}^{-p} \exp(i\pi pq/N). \tag{5.52}$$

That $\hat{R}$ is parity operator and it is worth noting that the phase space point operators have been defined on a lattice with $2N \times 2N$ points, but it has be shown that there are only N^2 independent phase space point operators on the set $G_N = \{\alpha = (q, p); 0 \leq q, p \leq N-1\}$. Here the wigner function has been obtained by $W(\alpha) = Tr(\hat{A}(\alpha)\hat{\rho})$.

In order to obtain truncated coherent state tomography relation in the FD Banach space we assume that relation (5.45) is a density matrix in ID Banach space. Also let

$$\rho^N = \sum_{\alpha \in G_N} \lambda_\alpha \mid \hat{N}_\alpha), \tag{5.53}$$

be a density matrix in FD Banach space which is obtained from truncating the ID Banach space, where $\hat{N}_\alpha = \frac{1}{\sqrt{N}} \hat{U}(\alpha)$. Using the properties of density matrix we have

$$\rho - \rho^N \geq 0, \tag{5.54}$$

which is comparison with semidefinite programming and using complementary slackness condition, we get

$$\hat{Z}(\rho - \rho^N) = 0 \;\text{ or }\; \hat{Z}(\rho - \lambda_\alpha \mid \hat{N}_\alpha)) = 0. \tag{5.55}$$

Similar to supperoperator formalism we obtain

$$\lambda_\alpha = (\hat{Q}_\alpha \mid \rho) = Tr[\rho \hat{Q}_\alpha^\dagger]. \tag{5.56}$$

The superoperator $\mathcal{G}$ is defined by $\mathcal{G} =\mid N_\alpha)(N_\alpha \mid= \frac{I}{N}$ is inevitable. Then Q_α operators are defined

$$Q_\alpha = \mathcal{G}^{-1} \mid N_\alpha) = \frac{1}{\sqrt{N}} U(\alpha). \tag{5.57}$$

Therefore we obtain the tomography formula in FD Hilbert space as the follow:

$$\hat{\rho}^N = 1/N \sum_{\alpha \in G_N} Tr(\hat{\rho} \hat{U}^\dagger(\alpha)) \hat{U}(\alpha) = 4N \sum_{\alpha \in G_N} W(\alpha) \hat{A}(\alpha). \tag{5.58}$$

If N is greater than or equal to the largest Fock state component of a given state, which by definition is our case, then the discrete probabilities (from the discrete Wigner) are proportional to the values of the continuous probability distribution in the discrete set of points (Miranowicz *et al.*, 2001). In this case the FD density matrix can be approximated as close as we want using ID methods, i.e., linear $\rho^N \to \rho$ in the $N \to \infty$ limit.

In the projection method we saw that, we can project any operator defined in ID space to FD space, then projected annihilation and creation operators in the FD space are given by

$$a_s = \sum_{n=1}^{s} \sqrt{n} |n-1\rangle\langle n|, \quad a_s^\dagger = \sum_{n=1}^{s} \sqrt{n} |n\rangle\langle n-1|. \tag{5.59}$$

Kuang *et al.* (1993, 1994) defined the normalized finite dimensional(FD) coherent states by truncating the Fock expansion of the conventional ID coherent states or equivalently by the action of the operator $\exp(\bar{\alpha}a^\dagger)$ (with proper normalization) on vacuum state. The state $|\bar{\alpha}>_{(s)}$, where $|\bar{\alpha}>_{(s)}= |\bar{\alpha}|\ \exp(i\phi)$, can be defined by its Fock expansion

$$|\bar{\alpha}>_{(s)}= \mathcal{N}_s\ \exp(\bar{\alpha}a_s^\dagger)|0\rangle = \sum_{n=0}^{s} b_n^{(s)}|n\rangle, \tag{5.60}$$

with the Poissonian superposition coefficients

$$b_n^{(s)} = \mathcal{N}_s \sum_{n=0}^{s} \frac{\bar{\alpha}^n}{\sqrt{n!}}, \tag{5.61}$$

normalized by

$$\mathcal{N}_s = \left(\sum_{n=0}^{s} \frac{\bar{\alpha}^{2n}}{n!}\right)^{-1/2}, \tag{5.62}$$

where these truncated CS can be defined by projection method. By definition, the truncated CS go over into the Glauber CS (1963a,b) in the limit of $s \mapsto \infty$.

The Wigner function for coherent pure states is given by Miranowicz *et al.* (1994, 2001)

$$W_s(n, \theta_m) = \frac{\mathcal{N}_s}{s+1}\left(\sum_{k=0}^{M} \frac{|\bar{\alpha}|^M}{\sqrt{k!(M-k)!}} \exp[i(2k-M)(\theta_m - \phi)] + \sum_{k=M+1}^{s} \frac{|\bar{\alpha}|^{M+s+1}}{\sqrt{k!(M-k+s+1)!}}\right), \tag{5.63}$$

where $M = 2n \bmod (s+1)$. If s is greater than or equal to the largest Fock state component of a given state, which by definition is our case, then the discrete probabilities (from the discrete Wigner) are proportional to the values of thec ontinuous phase probability distribution in the discrete set of points (Miranowicz *et al.*, 2001).

6. PROJECTION METHOD IN THE FINITE DIMENSIONAL AS A SEMIDEFINITE PROGRAMMING AND QUANTUM TOMOGRAPHY

6.1. Phase Tomography

One possible means of describing the phase of a quantum mechanical fields is in terms of the Pegg-Barnett hermitian phase operator Φ (Pegg and Barnett, 1989; Barnett and Pegg, 1989; Buzek *et al.*, 1999). This operator is defined in a finite (but arbitrary large) dimensional Hilbert space. In a $(s+1)$-dimensional

Hilbert space the phase state are defined as

$$\mid \theta\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\Phi} \mid n\rangle, \tag{6.64}$$

this Hilbert space is spanned by a complete orthonormal set of basis phase state $\mid \theta_m\rangle$, given by (6.64) with

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \quad m = 0, 1, \ldots, s, \tag{6.65}$$

where θ_0 is a reference phase. In terms of the state $\mid \theta_m\rangle$ the Hermitian phase operator is

$$\Phi_\theta = \sum_{m=0}^{s} \theta_m \mid \theta_m\rangle\langle\theta_m \mid . \tag{6.66}$$

From the definition of the phase state (6.64), we can express the projector $\mid \theta_m\rangle\langle\theta_m \mid$ in terms of the number state basis:

$$\mid \theta_m\rangle\langle\theta_m \mid = (s+1)^{-1} \sum_{n=0}^{s} \sum_{n'=0}^{s} e^{i(n'-n)\Phi} \mid n'\rangle\langle n \mid . \tag{6.67}$$

In this case Φ_θ is orthonormal then we can write the tomography using semidefinite programming.

At first we assume that

$$\rho = \int_\theta Tr(\rho\Phi_\theta)\Phi_\theta d\mu_\theta, \tag{6.68}$$

is a density matrix in Banach space. Also let

$$\rho' = \sum_\theta \lambda_\theta \mid \Phi_\theta), \tag{6.69}$$

be a density matrix in FD Banach space which is obtained from truncating the higher dimensional Banach space. Using the properties of density matrix we have

$$\rho - \rho' \geq 0, \tag{6.70}$$

which is comparison with semidefinite programming we get

$$F_0 = \rho, \quad F_\theta = \mid \Phi_\theta) \quad \text{and} \quad x_\theta = \lambda_\theta, \quad \text{for } \theta = \theta_0, \ldots, \theta_0 + 2\pi. \tag{6.71}$$

If we use the complementary slackness condition, and for a feasible $(\hat{Z}, \lambda_{\theta\max})$, for $\theta = \theta_0, \ldots, \theta_0 + 2\pi$, we have

$$\hat{Z}(\rho - \rho') = 0 \ \text{ or } \ \hat{Z}(\rho - \lambda_\theta \mid \Phi_\theta)) = 0. \tag{6.72}$$

Similar to superoperator formalism we obtain

$$\lambda_\theta = (\hat{\Phi}_\theta \mid \rho') = Tr[\rho' \Phi_\theta]. \tag{6.73}$$

Therefore we obtain the tomography formula in FD Hilbert space as the follow:

$$\rho' = \sum_\theta \mid \Phi_\theta)(\Phi_\theta \mid \rho') = \sum_\theta \mid \Phi_\theta) Tr[\rho' \hat{\Phi}_\theta]. \tag{6.74}$$

If we generalized it when θ is continuous, in this case we have

$$\rho = \int_{\theta_0}^{\theta_0 + 2\pi} Tr[\rho \Phi_\theta] \Phi_\theta d\theta. \tag{6.75}$$

Using (6.66) $Tr[\rho\Phi_\theta]$ obtain as follows

$$Tr[\rho\Phi_\theta] = Tr\left[\rho \sum_{m=0}^{s} \theta_m \mid \theta_m\rangle\langle\theta_m \mid \right] = 2\pi \sum_m \theta_m \frac{1}{s+1} P_{PB}(\theta)_m, \tag{6.76}$$

where P_{PB} is probability of measuring a particular value of phase and is normalized so that the integral of $P_{PB}(\Phi_\theta)$ over a 2π region of θ is equal to one.

$$P_{PB}(\Phi_\theta) = \frac{1}{2\pi} \sum_{n,n'=0}^{s} e^{i(n'-n)\phi} \langle n \mid \rho \mid n'\rangle = \langle\theta \mid \rho \mid \theta\rangle, \tag{6.77}$$

where thus obtained results are in agreement with those of already obtained by one of the authors in Buzek *et al.* (1999), Pegg and Barnett (1989), Barnett and Pegg (1989).

A very important subset of these states will be the physical partial phase states, of which the coherent state is a particular example. The phase states are themselves unphysical and so the best attempt at a physical phase measurement will only project the system into a physical partial phase state (Pegg and Barnett, 1989). In the following, we obtain a physical partial phase state tomography i.e., coherent spin states tomography.

6.2. Coherent Spin States Tomography

To reconstruct a mixed or pure quantum state of a spin s is possible through coherent states: its density matrix is fixed by the probabilities to measure the value s along $4s(s+1)$ appropriately chosen directions in space. Thus, after inverting the experimental data, the statistical operator is parameterized entirely by expectation values.

A coherent spin state $\mid n\rangle$ is associated to each point of the surface of the unit sphere.

$$\mid n\rangle \equiv \exp[-i\theta m(\phi) \cdot \hat{s}] \mid s, n_z\rangle, \tag{6.78}$$

where $m(\phi) = (-\sin\phi, \cos\phi, 0)$.

A stereographic projection of the surface of the sphere to the complex plane give the expansion of a coherent state (Weigert, 1999) as follows

$$| s, n\rangle = \frac{1}{(1+ | z |^2)^s} \sum_{k=0}^{2s} (2s/\!/k)^{1/2} z^k \mid s-k, n_z\rangle. \tag{6.79}$$

In order to show that the density matrix ρ of a spin s is determined unambiguously by appropriate measurement with a Stern-Gerlach apparatus one precedes as follows. Distribute $N_s = (2s+1)^2$ axes $\mid s, n\rangle$ with $1 \leq n \leq N_s$, over $(2s+1)$ cones about the z axis with different opening angles such that the set of the $(2s+1)$ directions on each cone is invariant under a rotation about z by an angle $\frac{2\pi}{(2s+1)}$.

An unnormalized statistical density operator is then fixed by measuring the N_s relative frequencies

$$p_n(n_n) = \langle n_n \mid \rho \mid n_n\rangle, \quad 1 \leq n \leq N_s, \tag{6.80}$$

that is, by the expectation values of the statistical operator $\hat{\rho}$ in the coherent states $\mid n_n\rangle$. You obtain N_s linear relations between probabilities $P_n(n_n)$ and the matrix elements of the density matrix with respect to the basis $|s-k, n_z\rangle$. This set of equations can be inverted by standard techniques if the directions n_n are chosen as described above. For a spin s, the projection operators

$$\mid Q_n) = \mid n_n\rangle\langle n_n \mid, \tag{6.81}$$

constitute thus a quorum Q. In general, a quorum is defined as a collection of (hermitian) operators having the property that their expectation values are sufficient to reconstruct the quantum state of the system at hand. $(Q^n \mid$ defined as the dual of the quorum (6.81):

$$\frac{1}{(2s+1)} \sum_{n=1}^{N_s} \sum_{n'=1}^{N_s} \mid Q_n)(Q^{n'} \mid = \delta_n^{n'}, \quad 1 \leq n, \quad n' \leq N_s. \tag{6.82}$$

Therefore, this coherent spin state introduced above is same as the phase state.

In order to obtain spin tomography relation in the finite dimensional Banach space we assume that

$$\rho = \int Tr(\rho \mid Q_m))(Q^m \mid d\mu_m, \tag{6.83}$$

is a density matrix in higher dimensional Banach space. Also let

$$\rho' = \sum_n \lambda_n \mid \hat{Q}^n), \tag{6.84}$$

be a density matrix in FD Banach space which is obtained from truncating the higher dimensional Banach space. Using the properties of density matrix we have

$$\rho - \rho' \geq 0, \tag{6.85}$$

which is comparison with semidefinite programming and using complementary slackness condition, we get

$$\hat{Z}(\rho - \rho') = 0 \text{ or } \hat{Z}(\rho - \lambda_n \mid \hat{Q}^n)) = 0. \tag{6.86}$$

Similar to supperoperator formalism we obtain

$$\lambda_n = (\hat{Q}_n \mid \rho) = \frac{1}{2s+1} Tr[\rho \hat{Q}^n] = \frac{1}{2s+1} P_n. \tag{6.87}$$

Therefore we obtain the tomography formula in FD Hilbert space as the follow:

$$\rho^s = \frac{1}{2s+1} \sum_{n=1}^{N_s} P_n Q^n, \tag{6.88}$$

where the coefficients P_n satisfy

$$0 \leq P_n \leq 1, \quad 1 \leq n \leq N_s. \tag{6.89}$$

The operators Q_n do even define an optimal quorum since exactly $(2s+1)^2$ numbers have to be determined experimentally which equals the number of free real parameters of the (unnormalized) hermitian density matrix $\hat{\rho}$. Thus obtained results are in agreement with those of already obtained by one of the authors in Buzek *et al.* (1999), Pegg and Barnett (1989), Barnett and Pegg (1989), and Weigert (1999).

It is important to note that, although each of the P_n is a probability, they do not sum up to unity:

$$0 < \sum_{n=1}^{N_s} P_n < (2s+1)^2. \tag{6.90}$$

This is due to the fact that they all refer to different orientations of the Stern–Gerlach apparatus, being thus associated with the measurement of incompatible observables,

$$[Q_n, Q_{n'}] \neq 0, \quad 1 \leq n, \quad n' \leq N_s, \tag{6.91}$$

since the scalar product $\langle n_n | n'_n \rangle$ of two coherent states is different from zero. The sum in (6.90) cannot take the value $(2s+1)^2$ since this would require a common eigenstate of all the operators Q_n which does not exist due to (6.91). By an appropriate choice of the directions n_n (all in the neighborhood of one single direction n_0, say), the sum can be arbitrarily close to $(2s+1)^2$ for states peaked about n_0. Similarly, the sum of all P_n cannot take on the value zero since this would require a vanishing density matrix which is impossible. If, however, considered as a sum of expectation values, there is no need for the numbers P_n to sum up to unity. Nevertheless, they are not completely independent when arising from a statistical operator: its normalization implies that

$$Tr[\rho^s] = Tr\left[\frac{1}{2s+1}\sum_{n=1}^{N_s} P_n Q^n\right] = 1, \tag{6.92}$$

turning one of the probabilities into a function of the $(2s+1)^2 - 1 = 4s(s+1)$ others, leaving us with the correct number of free real parameters needed to specify a density matrix (Weigert, 1999).

6.3. Qudit Tomography

We begin with the set of Hermitian generators of $SU(D)$. The generators, denoted by λ_j, are labeled by a Roman index taken from the middle of the alphabet, which takes on values $j = 1, \ldots, D^2 - 1$ (Rungta *et al.*). We represent the generators in an orthonormal basis $|a\rangle$, labeled by a Roman letter taken from the beginning of the alphabet, which takes on values $a = 1, \ldots, D$. With these conventions the generators are given by

$j = 1, \ldots, D-1$:

$$\lambda_j = \Gamma_a \equiv \frac{1}{\sqrt{a(a-1)}}\left(\sum_{b=1}^{a-1} |b\rangle\langle b| - (a-1)|a\rangle\langle a|\right), \quad 2 \le a \le D, \tag{6.93}$$

$j = D, \ldots, (D+2)(D-1)/2$:

$$\lambda_j = \Gamma_{ab}^{(+)} \equiv \frac{1}{\sqrt{2}}\left(|a\rangle\langle b| + |b\rangle\langle a|\right), \quad 1 \le a < b \le D, \tag{6.94}$$

$j = D(D+2)/2, \ldots, D^2 - 1$:

$$\lambda_j = \Gamma_{ab}^{(-)} \equiv \frac{-i}{\sqrt{2}}\left(|a\rangle\langle b| - |b\rangle\langle a|\right), \quad 1 \le a < b \le D. \tag{6.95}$$

In Eqs. (6.94) and (6.95), the Roman index j stands for the pair of Roman indices, ab, whereas in Eq. (6.93), it stands for a single Roman index a. The generators are traceless and satisfy

$$\lambda_j \lambda_k = \frac{1}{D}\delta_{jk} + d_{jkl}\lambda_l + i f_{jkl}\lambda_l. \tag{6.96}$$

Here and wherever it is convenient throughout this paper, we use the summation convention to indicate a sum on repeated indices. The coefficients f_{jkl}, the structure constants of the Lie group SU(D), are given by the commutators of the generators and are completely antisymmetric in the three indices. The coefficients d_{jkl} are given by the anti-commutators of the generators and are completely symmetric.

By supplementing the $D^2 - 1$ generators with the operator

$$\lambda_0 \equiv \frac{1}{\sqrt{D}} I, \tag{6.97}$$

where I is the unit operator, we obtain a Hermitian operator basis for the space of linear operators in the qudit Hilbert space. This is an orthonormal basis, satisfying

$$\mathrm{tr}(\lambda_\alpha \lambda_\beta) = \delta_{\alpha\beta}. \tag{6.98}$$

Here the Greek indices take on the values $0, \ldots, D^2 - 1$; throughout this paper, Greek indices take on D^2 or more values. Using this orthonormality relation, we can invert Eqs. (6.93)–(6.95) to give

$$|a\rangle\langle a| = \frac{I}{D} + \frac{1}{\sqrt{a(a-1)}}\left(-(a-1)\Gamma_a + \sum_{b=a+1}^{D} \Gamma_b\right), \tag{6.99}$$

$$|a\rangle\langle b| = \frac{1}{\sqrt{2}}\left(\Gamma_{ab}^{(+)} + i\Gamma_{ab}^{(-)}\right), \quad 1 \le a < b \le D, \tag{6.100}$$

$$|b\rangle\langle a| = \frac{1}{\sqrt{2}}\left(\Gamma_{ab}^{(+)} - i\Gamma_{ab}^{(-)}\right), \quad 1 \le a < b \le D. \tag{6.101}$$

Any qudit density operator can be expanded uniquely as

$$\rho = \frac{1}{D} c_\alpha \lambda_\alpha, \tag{6.102}$$

where the (real) expansion coefficients are given by

$$c_\alpha = D\mathrm{tr}(\rho\lambda_\alpha). \tag{6.103}$$

Normalization implies that $c_0 = \sqrt{D}$, so the density operator takes the form

$$\rho = \frac{1}{D}(I + c_j\lambda_j) = \frac{1}{D}(I + \vec{c}\cdot\vec{\lambda}). \tag{6.104}$$

Here $\vec{c} = c_j\vec{e}_j$ can be regarded as a vector in a $(D^2 - 1)$-dimensional real vector space, spanned by the orthonormal basis $\vec{e}_j$, and $\vec{\lambda} = \lambda_j\vec{e}_j$ is an operator-valued vector.

In order to treat discrete density operator representation for a qudit we introduce the superoperator formalism and SDP method. Consider a discrete set of projection operators (Rungta *et al.*) define in K dimensional Banach space

$$N_{\overrightarrow{n_\alpha}} = | \overrightarrow{n_\alpha}\rangle\langle\overrightarrow{n_\alpha} | = \frac{1}{D}(1 + \overrightarrow{\lambda}\cdot\overrightarrow{n_\alpha}), \quad \alpha = 1, \ldots, K. \tag{6.105}$$

The corresponding superoperator,

$$\mathcal{G} = K\left((D+1)\frac{|I)(I|}{D} + \mathcal{T}\right), \tag{6.106}$$

where, orthonormal eigenoperators of $\mathcal{G}$ are $\lambda_0 = I/\sqrt{D}$ and $\mathcal{T} = \sum_j \mid \lambda_j\rangle\langle\lambda_j \mid$.

We are now prepared to write the inverse of $\mathcal{G}$ with respect to the left-right action as

$$\mathcal{G}^{-1} = \frac{1}{K}\left(\frac{1}{D+1}\frac{|I)(I|}{D} + \mathcal{T}\right). \tag{6.107}$$

Thus the dual operators are given by

$$|Q_{n_\alpha}) = \mathcal{G}^{-1}|N_{n_\alpha}) = \frac{1}{K}\left(\mid N_\alpha) - \frac{\mid I)}{D+1}\right). \tag{6.108}$$

Using SDP method we get

$$F_0 = \frac{1}{D}(1 + c\cdot\lambda), \quad f_\alpha = \mid N_\alpha) \text{ and } \quad x_\alpha = \Lambda_\alpha \quad \text{for } \alpha = 1, \ldots, K. \tag{6.109}$$

From complementary slackness condition we have

$$\Lambda_\alpha = (Q_{n_\alpha} \mid \rho). \tag{6.110}$$

Therefore, tomography relation in FD Banach space can be represented in the form

$$\rho^K = \sum_{\alpha=1}^{K} \mid N_{n_\alpha})(Q_{n_\alpha} \mid \rho) = \sum_{\alpha=1}^{K} Tr[Q^{\dagger}_{n_\alpha}\rho]N_{n_\alpha} = \frac{1}{K}\sum_{\alpha=1}^{K} N_\alpha(I + Tr[N_{n_\alpha}\rho]). \tag{6.111}$$

A qubit is two-level system, for which $D = 2$. There is a one-to-one correspondence between the pure states of a qubit and the points on the unit sphere, or Bloch sphere (Schack and Caves, 1999). Any pure state of a qubit can be written in terms of the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$, as

$$N_{\overrightarrow{n}} = \mid \overrightarrow{n}\rangle\langle\overrightarrow{n} \mid$$

where $\overrightarrow{n} = (n_1; n_2; n_3)$ is a unit vector, and 1 denotes the unit matrix. An arbitrary state ρ, mixed or pure, of a qubit can be expressed as

$$\rho = \frac{1}{2}(1 + \overrightarrow{S}\cdot\overrightarrow{\sigma}) \tag{6.112}$$

where $0 \leq\mid S \mid\leq 1$.

In order to treat discrete density operator representation for a qubit we introduce the superoperator formalism and SDP method. Consider a discrete set of projection operators (Schack and Caves, 1999) in superoperator formalism

$$N_{\overrightarrow{n_\alpha}} = | \overrightarrow{n_\alpha}\rangle\langle\overrightarrow{n_\alpha} | = \frac{1}{2}(1 + \overrightarrow{\sigma} \cdot \overrightarrow{n_\alpha}), \quad \alpha = 1, \ldots, K. \tag{6.113}$$

The corresponding superoperator,

$$\mathcal{G} = \sum_{\alpha=1}^{K} | N_{\overrightarrow{n_\alpha}})(N_{\overrightarrow{n_\alpha}} | = \frac{1}{4}\Bigg[K \mid 1)(1 \mid + \sum_\alpha [\overrightarrow{n_\alpha} \cdot \mid \overrightarrow{\sigma})(1 \mid + \mid 1)(\overrightarrow{\sigma} \mid \cdot \overrightarrow{n_\alpha}]$$
$$+ \sum_{j,k} \mid \sigma_j)(\sigma_k \mid \sum_\alpha (n_\alpha)_j (n_\alpha)_k\Bigg], \tag{6.114}$$

generates dual-basis operators and expansion coefficients proportional to those for the continuous representation (Schack and Caves, 1999) if and only if

$$0 = \sum_\alpha \overrightarrow{n}_\alpha \tag{6.115}$$

$$\frac{1}{3}\delta_{jk} = \frac{1}{K}\sum_\alpha (n_\alpha)_j (n_\alpha)_k.$$

When these conditions are satisfied, the superoperator (6.114) simplifies to

$$\mathcal{G} = \frac{K}{4}\Bigg[\mid 1)(1 \mid + \frac{1}{3}\sum_j \mid \sigma_j)(\sigma_j \mid\Bigg], \tag{6.116}$$

with an inverse

$$\mathcal{G}^{-1} = \frac{1}{K}\Bigg[\mid 1)(1 \mid + 3\sum_j \mid \sigma_j)(\sigma_j \mid\Bigg], \tag{6.117}$$

which generates dual-basis operators

$$Q_{\overrightarrow{n}_\alpha} = \mathcal{G}^{-1} \mid N_{\overrightarrow{n}_\alpha}) = \frac{1}{K}(1 + 3\overrightarrow{\sigma}.\overrightarrow{n}_\alpha). \tag{6.118}$$

Then the density matrix in FD Banach space is given by (4.35). Using SDP method we get

$$F_0 = \frac{1}{2}(1 + S \cdot \sigma), \quad f_\alpha = \mid N_\alpha) \;\text{ and }\; x_\alpha = \lambda_\alpha \;\text{ for }\; \alpha = 1, \ldots, K. \tag{6.119}$$

From complementary slackness condition we have

$$\lambda_\alpha = (Q_{n_\alpha} \mid \rho) = Tr\left[\frac{1}{2K}(1 + 3\overrightarrow{\sigma} \cdot \overrightarrow{n}_\alpha)(1 + \overrightarrow{S} \cdot \overrightarrow{\sigma}) = \frac{1}{K}(1 + 3\overrightarrow{S} \cdot \overrightarrow{n}_\alpha))\right]. \tag{6.120}$$

Therefore, tomography relation (4.42) in finite dimensional Banach space can be represented in the form

$$\rho^K = \sum_{\alpha=1}^{K} \mid Q_\alpha)(N_\alpha \mid \rho) = \frac{1}{K}\sum_{\alpha=1}^{K} N_\alpha(1 + 3\overrightarrow{S}\cdot\overrightarrow{n}_\alpha), \tag{6.121}$$

For M qubits, we define the pure-product-state projector

$$N(\alpha) = N_{\alpha_1}\otimes\cdots\otimes N_{\alpha_M} = \frac{1}{2^M}(1 + s\cdot n_{\alpha_1})\otimes\cdots\otimes(1 + s\cdot n_{\alpha_M}), \tag{6.122}$$

and

$$Q(\alpha) = Q_{n_{\alpha_1}}\otimes\cdots\otimes Q_{n_{\alpha_M}} = \left(\frac{1}{4\pi}\right)^M (1 + 3s\cdot n_{\alpha_1})\otimes\cdots\otimes(1 + 3s\cdot n_{\alpha_M}), \tag{6.123}$$

where n stands for the collection of unit vectors $n_1, \ldots, n_M$. Any M-qubit density operator can be expanded as

$$\rho^K = \sum_{\alpha=1}^{K} \mid N_\alpha)(Q_\alpha \mid \rho) = \frac{1}{K^M}\sum_{\alpha_1,\ldots,\alpha_M=1}^{K} \times N_{\alpha_1}(1 + 3\overrightarrow{S}\cdot\overrightarrow{n}_{\alpha_1})\otimes\ldots N_{\alpha_M}(1 + 3\overrightarrow{S}\cdot\overrightarrow{n}_{\alpha_M}), \tag{6.124}$$

where thus obtained result is in agreement with those of already obtained by one of the authors in Buzek *et al*. (1999) and Schack and Caves (1999).

7. CONCLUSION

Using the elegant method of convex semidefinite optimization method and superoperator formalism, we have been able to obtain the quantum tomography in finite dimensional representation for some set of mixed density matrices. In this method we have been able to obtain truncated coherent states tomography, finite phase tomography and coherent spin state tomography, qudit, N-qubit quantum tomography, where these results that obtained are in agreement with those of Schack and Caves (1999), Pegg and Barnett (1989), Barnett and Pegg (1989), Buzek *et al*. (1999), Weigert (1999).

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